A relaxation of the Directed Disjoint Paths problem: a global congestion metric helps

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XP and FPT

Problem $\mathcal{P}$ of size n with a parameter k:

- $\mathcal{P} \in \mathsf{XP} \implies \mathcal{P}$ can be solved in $\mathcal{O}\left(g(k) \cdot n^{f(k)}\right)$.

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- k-Clique $\in$ XP and is W[1]-hard.

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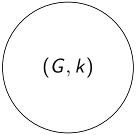

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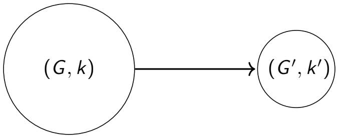

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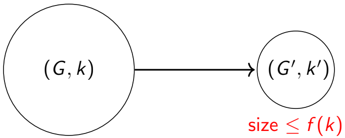

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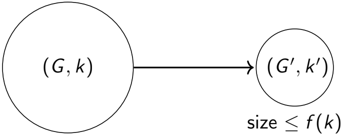

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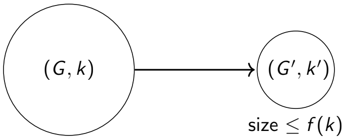

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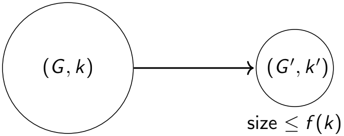

Theorem

A parameterized problem $\mathcal{P}$ has a kernel $\iff$ $\mathcal{P}$ is FPT.

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In a digraph D:

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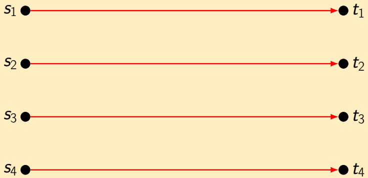
Pairwise vertex-disjoint paths.

$\Longrightarrow$ k-Directed Disjoint Paths (k-DDP).



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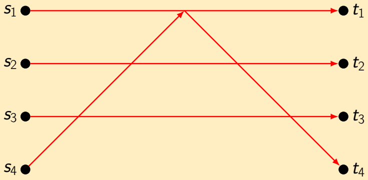

$c = 2$ in the example.

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k-DDP

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The directed subgraph homeomorphism problem
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A. Slivkins
Parameterized Tractability of Edge-Disjoint Paths on Directed Acyclic Graphs
SIAM Journal on Discrete Mathematics 24.1, 2010

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T. Johnson and N. Robertson and P.D. Seymour and R. Thomas,
Directed tree-width
Journal of Combinatorial Theory, Series B, Volume 82, Issue 1, 2001, Pages 138-154

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Extension of Slivkins reduction, and improved upon.

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- k-DDP is FPT in planar digraphs.

Cygan, M., Marx, D., Pilipczuk, M. and Pilipczuk, M.
The planar directed k-vertex-disjoint paths problem is fixed-parameter tractable
In Proc. of the IEEE 54th Annual Symposium on Foundations of Computer Science (FOCS), 2013

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Edwards, K., Muzi, I. and Wollan, P.
Half-integral linkages in highly connected directed graphs
In Proc. of the 25th Annual European Symposium on Algorithms (ESA), 2017

- k-DDP is FPT in planar digraphs.
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- DDPC is XP with param $d = k - c$ in DAGs.

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Summary of positive results

- Congestion c.
- $d = k - c$.

Version \ Digraph	DAG	dtw $\leq w$	Planar	Strong
DDP	XP on k	XP on $k + w$	FPT on k	$- - -$
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K. Kawarabayashi and S. Kreutzer.
The Directed Grid Theorem
Proceedings of the Forty-seventh Annual ACM Symposium on Theory of Computing

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We bring positive news for *general* digraphs.

The Disjoin Enough Paths problem

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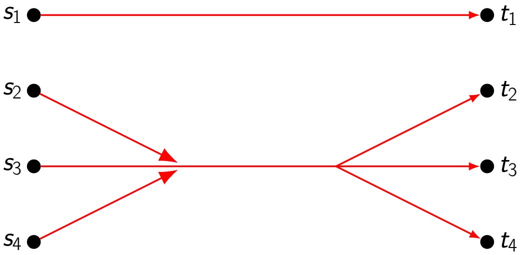

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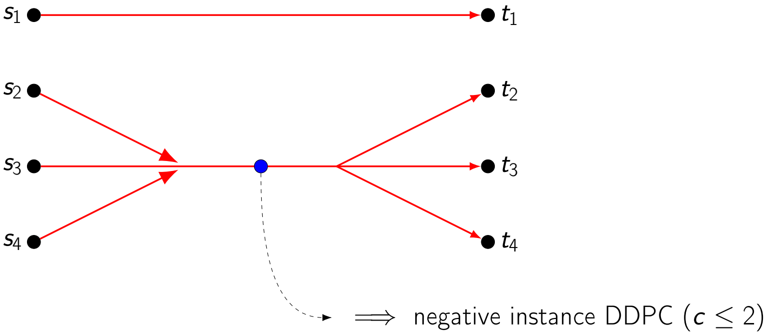

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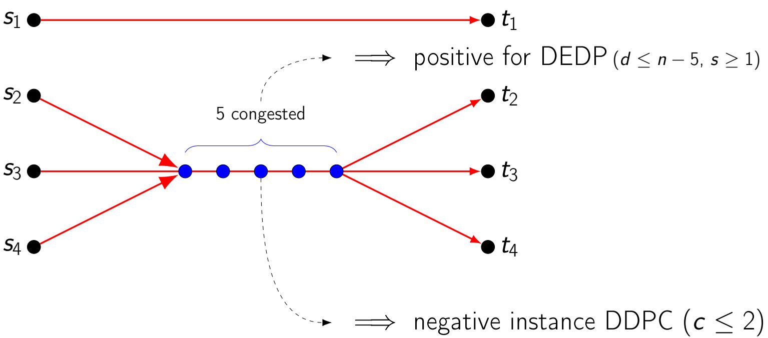

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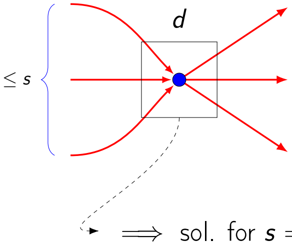

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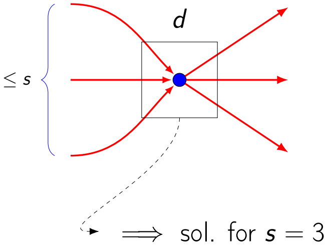

- $d = n, s = 1 \implies$ paths **d**isjoint in $V(D) \implies$ DDP.
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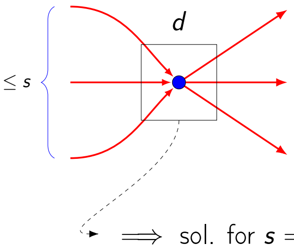

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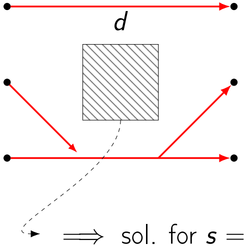

$\Longrightarrow$ sol. for $s = 0$

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- $s = 0 \implies$ paths avoiding d vertices $\implies$ Steiner Network.

Summary of results

for $0 < \alpha \leq 1$:

Hardness results

- NP-complete for $k \geq 3$, $d = n^{\alpha}$, fixed $s \geq 1$.
- W[1]-hard in DAGs with param. k, $d = n^{\alpha}$, fixed $s \geq 1$.
- W[1]-hard in DAGs with param. d, fixed $s \geq 0$.

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Positive results (all for *general digraphs*)

- XP with params. k and $\mathrm{dtw}(D)$.
- XP with params. d and s.
- FPT with params. k, d and s.

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- $D \longrightarrow$ Input digraph
- $I \longrightarrow$ Set of requests $\{(s_1, t_1), \ldots, (s_k, t_k)\}$
- $k \longrightarrow$ $k = |I|$
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- $s \longrightarrow$ Local congestion metric

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To simplify the presentation, we consider that:

Goal: Find collection of paths *satisfying* I s.t. at least d vertices occur in at most s paths of the collection.

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DEDP instances

Instance (D, I, k, d, s) where:

- $D \longrightarrow$ Input digraph
- $I \longrightarrow$ Set of requests $\{(s_1, t_1), \ldots, (s_k, t_k)\}$
- $k \longrightarrow$ $k = |I|$
- $d \longrightarrow$ Global congestion metric
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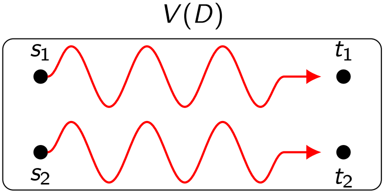

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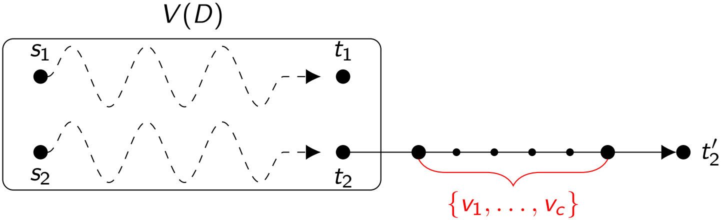

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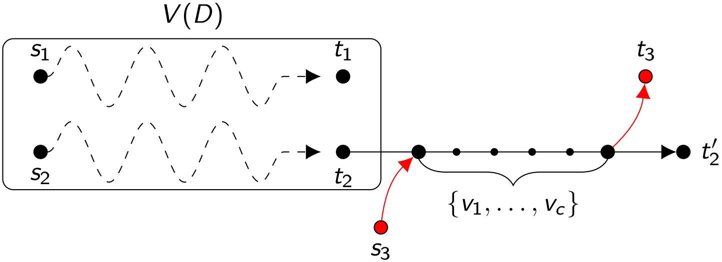

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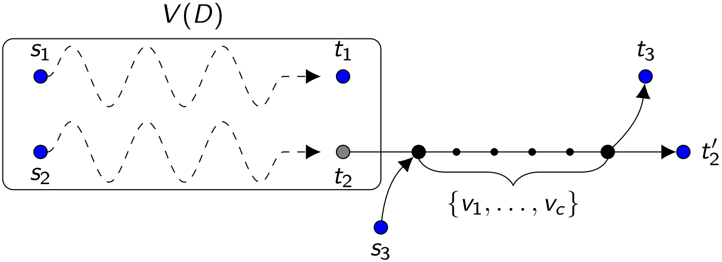

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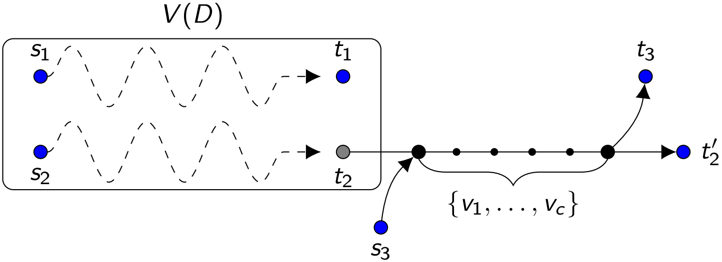

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- $s_3 \to t_3$ path $\implies$ c congested vertices $\implies$ $V(D) = \{\text{disjoint part}\}$.

Kernelization algorithm overview

Instance (D, I, k, d) ($s = 1$ omitted).

Definition (Congested vertices)

A vertex v is *congested* if v *blocks* ≥ 2 requests. That is, $\exists i, j$ s.t. $i \neq j$ and there is no path from s_i to t_i and no path from s_j to t_j in $D \setminus \{v\}$.

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Step 4: Use Steps 2 and 3 to solve *clean* instances with $n \geq f(k, d)$.

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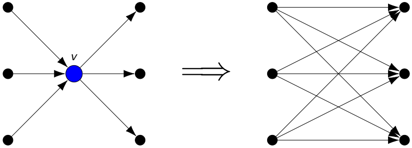

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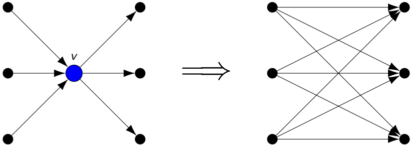

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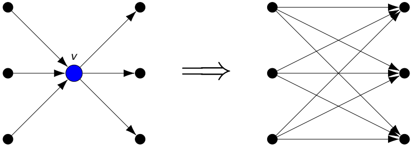

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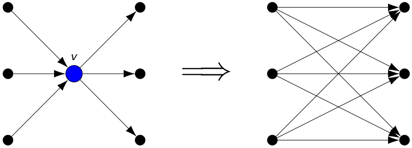

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Step 1

Bypass all *congested* vertices to generate *clean* instance.

Clean instance $k \to k-1$

- $n = |V(D) - \{\text{sources}\} - \{\text{terminals}\}|$.
- Clean instance, $s = 1$, $d^-(s_i) = d^+(t_i) = 0$, all terminals distinct.

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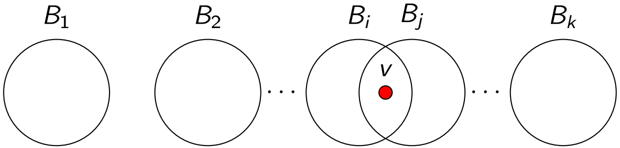

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$\implies \sum_{i \in [k]} |B_i| \geq n + 1 \implies \exists$ *congested* $v \implies$ ctd.

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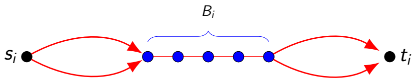

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- $\exists |B_i| \leq \frac{n}{k}$.

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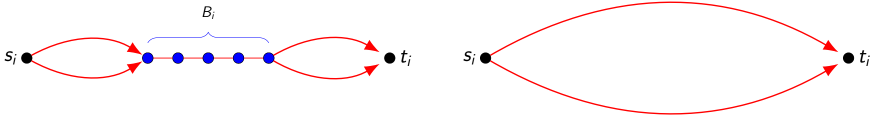

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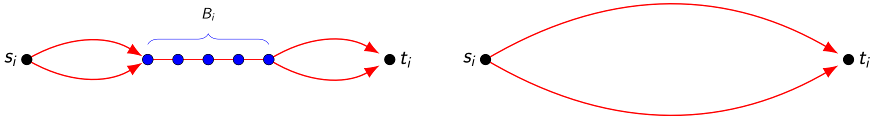

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- $\exists |B_i| \leq \frac{n}{k}$.
  - Bypass $B_i \implies 2$ internally disjoint $s_i \to t_i$ paths.
  - Take shortest P_i from $s_i \to t_i$.
- $|V(D/B_i)| \geq \frac{n(k-1)}{k} \implies |V(D/(B_i \cup P_i))| \geq \frac{n(k-1)}{2k}$.

Large clean instances

Lemma (Step 2 (Iteration))

(D, I, k, d) clean $\implies \exists P_i$ from $s_i \to t_i$ s.t. $|V(D/(B_i \cup P_i)| \geq \frac{n(k-1)}{2k}$.

Lemma (Step 3 (Base))

(D, I, k, d) clean, $k = 2$, $n \geq 4d \implies$ positive instance, solution in polynomial time.

- For large enough n :
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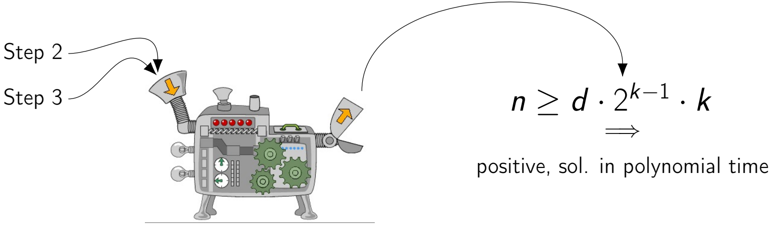

Math Machine

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- Our result $\implies$ kernel for Steiner Network with params. k and $d = n - c$, where c is the size of the solution.

Disjoint Enough Paths:

Input: *Requests* $I = \{(s_1, t_1), \ldots, (s_k, t_k)\}$, integers d and s.

Output: $\mathcal{P} = \{P_1, \ldots, P_k\}$ *satisfying* I s.t. $\geq d$ vertices occurring in $\leq s$ *paths*.

Parameters	Generalizes
$d = n$, $s = 1$	Disjoint Paths
$d = n$, $s \geq 2$	Disjoint Paths with Congestion
$d \geq 1$, $s = 0$	Steiner Network

k	d	s	dtw	Complexity
fixed ≥ 3	n^α	fixed ≥ 1	—	NP-complete
parameter	n^α	fixed ≥ 1	0	W[1]-hard
input	parameter	fixed ≥ 0	—	W[1]-hard
parameter	—	—	parameter	XP
input	parameter	parameter	—	XP
parameter	parameter	parameter	—	FPT

- Positive results for general digraphs.
- Kernel size: $d \cdot 2^{k-s} \cdot \binom{k}{s} + 2k$.
- Open: Poly-kernel. Negative answer for $s = 0$ suffices.
- Consequence for Steiner Network particularly interesting.